

GraphLib

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# Chapter 1

## Introduction

### 1.1 Over category

**Proposition 1.1.1** (Sliced adjoint functors). *If  $a : F \vdash G$  is an adjunction between  $F : C \rightarrow D$  and  $G : D \rightarrow C$  and  $X : C$ , then there is an adjunction between  $F/X : C/X \rightarrow D/F(X)$  and  $G/X : D/F(X) \rightarrow C/X$ .*

*Proof.* See <https://ncatlab.org/nlab/show/sliced+adjoint+functors+-+section>. □

**Proposition 1.1.2** (Limit-preserving functors lift to over categories). *Let  $J$  be a shape (i.e. a category). Let  $\tilde{J}$  denote the category which is the same as  $J$ , but has an extra object  $*$  which is terminal. If  $F : C \rightarrow D$  is a functor preserving limits of shape  $\tilde{J}$ , then the obvious functor  $C/X \rightarrow D/F(X)$  preserves limits of shape  $J$ .*

*Proof.* Extend a functor  $K : J \rightarrow C/X$  to a functor  $\tilde{K} : \tilde{J} \rightarrow C$ , by letting  $\tilde{K}(*) = X$ . □

**Proposition 1.1.3** (Essential image of a sliced functor). *If  $F : C \rightarrow D$  is a full functor between cartesian-monoidal categories, then  $F/X : C/X \rightarrow D/F(X)$  has the same essential image as  $F$ .*

*Proof.* Transfer all diagrams. □

# Chapter 2

## Preliminaries

### 2.1 Tensor Product

**Lemma 2.1.1** (The tensor product of linearly independent families). *Let  $R$  be a domain and  $M, N$  two  $R$ -semimodules. If  $f$  and  $g$  are linearly independent families of points in  $M$  and  $N$ , then  $(i, j) \mapsto fi \otimes gj$  is a linearly independent family of points in  $M \otimes N$ .*

*Proof.* We will prove the equivalent statement:

Let  $P, Q$  be two free  $R$  modules,  $f : P \rightarrow M$  and  $g : Q \rightarrow N$  be two  $R$ -linear injective maps. Then  $f \otimes g : P \otimes_R Q \rightarrow M \otimes_R N$  is injective.

Let  $K$  be the field of fractions of  $R$ .

The map

$$P \otimes_R Q \rightarrow (K \otimes_R P) \otimes_R (K \otimes_R Q) = (K \otimes_R P) \otimes_K (K \otimes_R Q)$$

is injective because  $R \rightarrow K$  is injective and all the modules involved are flat. The map

$$(K \otimes_R P) \otimes_K (K \otimes_R Q) \rightarrow (K \otimes_R M) \otimes_K (K \otimes_R N)$$

is injective because all the modules involved are  $K$ -flat (as  $K$  is a field).

$P \otimes_R Q \rightarrow M \otimes_R N$  is now a factor of the composition of the two injections above, and is thus injective.  $\square$

# Bibliography